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Set theory.

Q.1. State Zorn's Lemma and prove that Zorn's Lemma implies axiom of choice.

Soln: Statement: Let X is a non-empty partially ordered set and in it each chain has an upper bound then X has a maximal element.

proof. Let X is a nonempty partially ordered set in which each chain has an upper bound.

We take a non-empty set Z of subsets of X. It has the following two properties every subset of each member of Z is in Z and the union of each chain in Z lies in Z.

It is clear that $\phi \in Z$. We shall first prove that there is a maximal member in Z. Therefore we shall see that X has a maximal member.

We assume that for any nonempty set X there is a function

$$f: P(X) - \{\phi\} \to X$$

such that for each $A \in P(X) - \{\phi\}$, $f(A) \in A$.

Then, X will have a member which is a choice function.

For, each $B \in Z$, we define

$$\bar{B} = \{x / x \in X \text{ and } B \cup \{x\} \in Z\}.$$

We also define $\bar{f}: Z \to Z$, where

$$\bar{f}(B) = \begin{cases} B, & \text{if } \bar{B} - B = \phi, \\ B \cup \{f(\bar{B} - B)\}, & \text{if } \bar{B} - B \neq \phi. \end{cases}$$

It is clear that $B \subseteq \bar{f}(B)$ and when $B \neq \bar{f}(B)$ then $\bar{f}(B) - B$ will have just one element.

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Set Theory

Q.2. Explain the concept of an ordinal number.

SoLn: Let α is any set, α is called an ordinal number when α has the following properties:

(I) α is well-ordered by the relation $\in$ and this relation is irreflexive.

(II) If $\beta \in \alpha$, then $\beta \subseteq \alpha$.

Briefly an ordinal number is simply called an ordinal.

If R is a well-ordering relation in X, then R is a linear ordering.

We see that in α, $\in$ is an irreflexive linear order.

From definition, it is clear that the empty set ϕ has vacuously both the properties. Hence, ϕ is an ordinal.

If $\alpha \neq \phi$, then first element of α is ϕ.

If ordinal β is a proper subset of ordinal α, then $\beta \in \alpha$

Q.3. Prove that every set of ordinals is well-ordered.

Proof: Let A is a set of ordinals. We know that if α and β are two ordinals then $\alpha = \beta \Rightarrow \alpha \in \beta$ or $\beta \in \alpha$. A is ordered by $\in$, which is an irreflexive relation.

We shall prove that A is well-ordered by $\in$. Let $B \subseteq A$ and $B \neq \phi$. Then B has an element α. If for each $\beta \in B$, $\alpha = \beta$ or $\alpha \in \beta$, then α itself is the first element of B. Otherwise, there will be a $\beta \in B$ for which $\beta \in \alpha$. Then $\beta \in \alpha \cap B$. Hence $\alpha \cap B \neq \phi$ and $\alpha \cap B \subseteq \alpha$. Since, α is well-ordered under $\in$, $\alpha \cap B$ will have a first element γ. This γ will be the first element of B, for if possible, Let $\delta \in B$ for which $\delta \in \gamma$. Then since, $\gamma \in \alpha$ hence $\delta \in \alpha$. But, then $\delta \in \alpha \cap B$ and $\delta \in \gamma \in \alpha \cap B$ which is contradictory.

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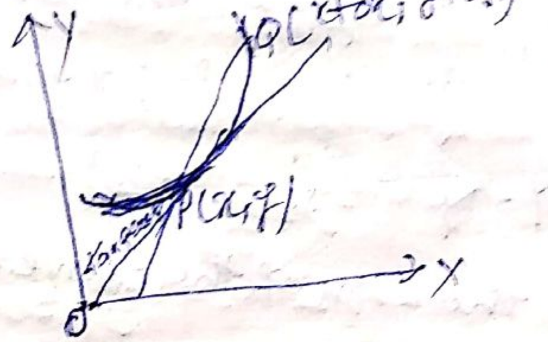
Tangents and Normals Date-23-04-2020

Q.1 Find the eqn to the tangent to the curve $y = f(x)$ at (x, y) is $Y - y = \frac{dy}{dx}(X - x)$ and that to the curve $f(x, y) = 0$ is
$(X - x) f_x + (Y - y) f_y = 0$

Soln.



Let P and Q be two points on the curve. By the tangent to the curve at P we mean the straight line which is the limiting position to which secant PQ tends as Q tends to P travelling along the curve.

Let the co-ordinates of the points P and Q be respectively (x, y) and $(x+\delta x, y+\delta y)$.

Let the eqn to the curve be $y = f(x)$
Then, $y + \delta y = f(x+\delta x)$ when $(x+\delta x, y+\delta y)$ is on the curve.

$$\therefore \frac{\delta y}{\delta x} = \frac{f(x+\delta x) - f(x)}{\delta x}$$

Let (X, Y) be the current co-ordinates. Then the eqn to the straight line PQ joining the points $P(x, y)$ and $Q(x+\delta x, y+\delta y)$ is

$$\frac{Y-y}{X-x} = \frac{(y+\delta y)-y}{(x+\delta x)-x} \Rightarrow \frac{Y-y}{X-x} = \frac{\delta y}{\delta x} = \frac{(y+\delta y)-f(x)}{\delta x}$$

Now, let Q tend to P. Then $\delta x \to 0$ and the straight line PQ will become the tangent to the curve at P(x, y)

$$\therefore \frac{Y-y}{X-x} = \lim_{\delta x \to 0} \frac{f(x+\delta x)-f(x)}{\delta x}$$

$$= f'(x) = \frac{dy}{dx}$$

Hence, the eqn to the tangent to the curve $y = f(x)$ at $P(x,y)$ is

$$Y-y = \frac{dy}{dx}(X-x) \quad \text{———} \quad (\mathrm{I})$$

Now, if the eqn of the curve be given in the form $f(x,y)=0$

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$

Substituting the value of $\frac{dy}{dx}$ in (I), we get

$$Y-y = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}(X-x)$$

$$\Rightarrow (X-x)\frac{\partial f}{\partial x} + (Y-y)\frac{\partial f}{\partial y} = 0$$

$$\Rightarrow (X-x)f_x + (Y-y)f_y = 0$$

Which is the eqn of the tangent to the curve $f(x,y)=0$ at $P(x,y)$